Zero bias transformation and asymptotic expansions

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Abstract

We apply the zero bias transformation to deduce a recursive asymptotic expansion formula for expectation of functions of sum of independent random variables in terms of normal expectations and we discuss the remainder term estimations.

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1 Introduction

Zero bias transformation has been introduced by Goldstein and Reinert [7] in the framework of Stein's method. By the fundamental works of Stein [15, 16], we know that a random variable (r.v.) Z with mean zero follows the normal distribution $N(0,\sigma^2)$ if and only if $\mathbb{E}[Zf(Z)] = \sigma^2\mathbb{E}[f'(Z)]$ for any Borel function f such that both sides of the equality are well defined. More generally, for any r.v. X with mean zero and finite variance $\sigma^2 > 0$, a r.v. X^* is said to have the zero biased distribution of X if the equality

$$\mathbb{E}[Xf(X)] = \sigma^2\mathbb{E}[f'(X^*)] \tag{1}$$

holds for any differentiable function f such that (1) is well defined. So combined with the *Stein's equation* $xf(x) - \sigma^2 f'(x) = h(x) - \Phi_\sigma(h)$ where h is a given function and $\Phi_\sigma(h)$ denotes the expectation of h under $N(0,\sigma^2)$, we have

$$\mathbb{E}[h(X)] - \Phi_\sigma(h) = \mathbb{E}[Xf_h(X) - \sigma^2 f_h'(X)] = \sigma^2\mathbb{E}[f_h'(X^*) - f_h'(X)]$$

where f_h is the solution of Stein's equation given by

$$f_h := \frac{1}{\sigma^2\phi_\sigma(x)}\int_x^\infty (h(t) - \Phi_\sigma(h))\phi_\sigma(t)\,dt. \tag{2}$$

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An important remark is that X^* need not be independent of X ([7], see also [6]). In fact, let $W = X_1 + \cdots + X_n$ be sum of independent mean zero random variables, Goldstein and Reinert proposed the construction $W^* := W^{(I)} + X_I^*$ where, for any $i \in \{1, \cdots, n\}$, $W^{(i)} := W - X_i$ and X_i^* is independent of $W^{(i)}$, and I is a random index valued in $\{1, \cdots, n\}$ which is independent of $(X_1, \cdots, X_n, X_1^*, \cdots, X_n^*)$ and satisfies $\mathbb{P}(I = i) = \sigma_i^2/\sigma_W^2$ with σ_i^2 being the variance of X_i and σ_W^2 that of W. We observe that the above construction of zero bias transformation is quite similar to Lindeberg method except that, in zero bias transformation, we consider an average of punctual substitutions of X_i by X_i^*; while in Lindeberg method, we substitute progressively X_i by central normal distribution with the same variance.

The asymptotic expansion of expectations of the form $\mathbb{E}[h(W)]$ is a classical topic in central limit theorems. Using Stein's method, Barbour [1, 2] has obtained a full expansion of $\mathbb{E}[h(W)]$ for sufficiently regular function h. Compared to the classical Edgeworth expansion (see [13, ChapV], also [14]), the results of [1] do not require the distribution of X_i to be smooth; however, as a price paid, we need some suitable regularity conditions on the function h. The result of [1] can also be compared to those in [9, 8] using Fourier transform. The key point of Barbour's method is a Taylor type formula with cumulant coefficients, which allows to write the difference $\mathbb{E}[Wf(W)] - \sigma_W^2\mathbb{E}[f'(W)]$ as a series which involves cumulants of order ≥ 3 and to iterate the procedure of replacing W-expectations by normal expectations until the desired order. It has been pointed out in [14] that the key formula of Barbour can also be obtained by Fourier transform.

Zero bias transformation have been used in [5] to obtain a first order correction term for the normal approximation of $\mathbb{E}[h(W)]$, where the motivation was to find a rapid numerical method for large-sized credit derivatives. The function of interest is the so-called *call function* in finance: $h(x) = (x - k)_+$ where k is a real number. Since such h is only absolutely continuous, the function f_h is not regular enough to have the third order derivative. To achieve the estimation, the authors have used a conditional expectation technique, together with a concentration inequality due to Chen and Shao [3, 4].

The main difficulty in generalizing the result in [5] to obtain a full expansion of $\mathbb{E}[h(W)]$ is that W and $W^* - W$ are not independent. In fact, if we consider the Taylor expansion of $f_h'(W^*)$ at W and then apply the expectation, there appear terms of the form $\mathbb{E}[f_h^{(l)}(W)(W^* - W)^k]$, where $f^{(l)}$ denotes the l^{th}-order derivative of f. For the first order expansion in [5], the conditional expectation argument allows us to replace $\mathbb{E}[f_h''(W)(W^* - W)]$ by $\mathbb{E}[f_h''(W)]\mathbb{E}[W^* - W]$ and put the covariance in the error term. However, in higher order expansion, the error term could no longer contain such covariances. An alternative way is to consider the Taylor expansion of $\mathbb{E}[f_h'(W^*) - f_h'(W)]$ at $W^{(i)}$. As X_i^* is independent of $W^{(i)}$, there is no crossing term. However, the expectations of the form $\mathbb{E}[f_h^{(l)}(W^{(i)})]$ appear, which make it difficult to apply the recurrence procedure. To overcome this difficulty, we propose a so-called reverse Taylor formula which enables us to replace $\mathbb{E}[f_h^{(l)}(W^{(i)})]$ by expectation of functions of W, up to an error term.

Let N be a positive integer, X and Y be two independent random variables such that Y has up to N^{th} order moments, and f be an N^{th} order differentiable function such that $f^{(k)}(X)$ and $f^{(k)}(X+Y)$ are integrable for any $k = 0, \cdots, N$. We define the notation $m_Y^{(k)} := \mathbb{E}[Y^k]/k!$. Denote by $\delta_N(f, X, Y)$ the error term in the N^{th} order Taylor expansion of $\mathbb{E}[f(X+Y)]$. Namely,

$$\delta_N(f, X, Y) := \mathbb{E}[f(X+Y)] - \sum_{k=0}^{N} m_Y^{(k)} \mathbb{E}[f^{(k)}(X)]. \tag{3}$$

Recall that for any $N \geq 1$,

$$\delta_N(f, X, Y) = \frac{1}{(N-1)!} \int_0^1 (1-t)^{N-1} \mathbb{E}\Big[\big(f^{(N)}(X+tY) - f^{(N)}(X)\big) Y^N\Big]\, dt \tag{4}$$

provided that the term on the right side is well defined. This is a consequence of the classical Taylor formula in its integral form (e.g. [12]).

The so-called reverse Taylor formula gives an expansion of $\mathbb{E}[f(X)]$ in terms of expectations of functions of $X+Y$ and of moments of Y. We would like to note that, in the expansion formula (5), the variables $X+Y$ and Y are not independent. We specify some notation and conventions. First of all, $\mathbb{N}_* := \mathbb{N} \setminus \{0\}$ denotes the set of strictly positive integers. For any integer $d \geq 1$ and any $\mathbf{J} = (j_l)_{l=1}^d \in \mathbb{N}_*^d$, $|\mathbf{J}|$ is defined as $j_1 + \cdots + j_d$, and $m_Y^{(\mathbf{J})} := m_Y^{(j_1)} \cdots m_Y^{(j_d)}$. By convention, $\mathbb{N}_*^0$ denotes the set $\{\emptyset\}$ of the empty vector, $|\emptyset| = 0$ and $m_Y^{(\emptyset)} = 1$.

Proposition 1.1 (Reverse Taylor formula) *With the above notation, the equality*

$$\mathbb{E}[f(X)] = \sum_{d \geq 0} (-1)^d \sum_{\mathbf{J} \in \mathbb{N}_*^d,\, |\mathbf{J}| \leq N} m_Y^{(\mathbf{J})} \mathbb{E}[f^{(|\mathbf{J}|)}(X+Y)] + \varepsilon_N(f, X, Y) \tag{5}$$

holds, where $\varepsilon_N(f, X, Y)$ is defined as

$$\varepsilon_N(f, X, Y) = -\sum_{d \geq 0} (-1)^d \sum_{\mathbf{J} \in \mathbb{N}_*^d,\, |\mathbf{J}| \leq N} m_Y^{(\mathbf{J})} \delta_{N-|\mathbf{J}|}(f^{(|\mathbf{J}|)}, X, Y). \tag{6}$$

The main result of this paper is an expansion formula for the sum of independent random variables. We present below its formal form without giving precise conditions on the function and on the summand variables (this will be done in Section 3). The methodology appeals to the zero bias transformation. From now on, we consider a family of independent random variables X_i $(i = 1, \cdots, n)$, with mean zero and finite variance $\sigma_i^2 > 0$. Let $W = X_1 + \cdots + X_n$ and $\sigma_W^2 = \mathrm{Var}(W)$. Denote by X_i^* a random variable which follows the zero-biased distribution of X_i and which is independent of $W^{(i)} := W - X_i$.

Theorem 1.2 *Assume that $X_1, \cdots, X_n$ and the function h are sufficiently good (in a sense that we shall precise later). Then, for any integer $N \geq 0$, $\mathbb{E}[h(W)]$ can be written as the sum of two terms $C_N(h)$ and $e_N(h)$, with $C_0(h) = \Phi_{\sigma_W}(h)$ and $e_0(h) = \mathbb{E}[h(W)] - \Phi_{\sigma_W}(h)$, and recursively for $N \geq 1$,*

$$C_N(h) = C_0(h) + \sum_{i=1}^{n} \sigma_i^2 \sum_{d \geq 1} (-1)^{d-1} \sum_{\mathbf{J} \in \mathbb{N}_*^d,\, |\mathbf{J}| \leq N} m_{X_i}^{(\mathbf{J}^\circ)} \big(m_{X_i^*}^{(\mathbf{J}^\dagger)} - m_{X_i}^{(\mathbf{J}^\dagger)}\big) C_{N-|\mathbf{J}|}(f_h^{(|\mathbf{J}|+1)}), \tag{7}$$

$$
(8) \quad \begin{aligned} e_N(h) = \sum_{i=1}^n \sigma_i^2 \Big[\sum_{d\geq 1} (-1)^{d-1} \sum_{\mathbf{J}\in\mathbb{N}_*^d,\, |\mathbf{J}|\leq N} m_{X_i}^{(\mathbf{J}^\circ)} \big(m_{X_i^*}^{(\mathbf{J}^\dagger)} - m_{X_i}^{(\mathbf{J}^\dagger)}\big) e_{N-|\mathbf{J}|}(f_h^{(|\mathbf{J}|+1)}) \\ + \sum_{k=0}^N \varepsilon_{N-k}(f_h^{(k+1)}, W^{(i)}, X_i) m_{X_i^*}^{(k)} + \delta_N(f_h', W^{(i)}, X_i^*) \Big], \end{aligned}
$$

where for any integer $d \geq 1$, and any $\mathbf{J} \in \mathbb{N}_*^d$, $\mathbf{J}^\dagger \in \mathbb{N}_*$ denotes the last coordinate of $\mathbf{J}$, and $\mathbf{J}^\circ$ denotes the element in $\mathbb{N}_*^{d-1}$ obtained from $\mathbf{J}$ by omitting the last coordinate.

In view of the classical formula relating the cumulants and moments, our principal term $C_N(h)$ is similar to that obtained by Barbour. Note that in $C_N(h)$, there appear normal expectations of iteration of operators which are of the form $g \mapsto f_g^{(l)}$ acting on h. As pointed out by Barbour [1, p.294], such expectation can be expressed as expectation of h multiplied by a Hermite polynomial.

The proof of the equality $\mathbb{E}[h(W)] = C_N(h) + e_N(h)$ is based on the reverse Taylor formula and the zero bias transformation. It is important to precise the conditions under which all terms in the formal expansion are well defined. Moreover, we also need to show that $e_N(h)$ is "small' enough as an error term. In our results, the error term $e_N(h)$ is expressed in a recursive way so that it is actually a linear combination of remainders of Taylor and reverse Taylor formulas and can be thus estimated. A key ingredient in the estimation is a concentration inequality which provides upper bound for $\mathbb{P}(a \leq W \leq b)$ involving exponent ≤ 1 of the interval length $(b-a)$, i.e. $(b-a)^\alpha$ with $0 < \alpha \leq 1$. This allows to us to obtain, under relatively mild moment conditions on X_i's than those in [1], estimations for the Taylor and reverse Taylor remainders. For example, as a consequence of Theorem 1.2 and the remainder estimations, we recover a classical result, initially obtained by using Fourier transform, asserting that if $X_1, \cdots, X_n$ are i.i.d. random variables with mean zero and variance $\sigma^2 > 0$, which admit $(2+\alpha)^{\text{th}}$ order moments, then the law of $(X_1 + \cdots + X_n)/\sqrt{n}$ converges to $N(0, \sigma^2)$ and that the convergence speed is of order $(1/\sqrt{n})^\alpha$.

The rest of the paper is organized as follows. We firstly prove the reverse Taylor formula and the formal expansion in Section 2. In Section 3, we introduce the admissible function space and discuss the conditions on h and on X_i's; this is inspired by ideas in [1] and we can in addition include some more irregular functions. We then restate the main expansion result in this context. Section 4 is devoted to error estimations. Finally, some technical proofs are left in Appendix.

2 Reverse Taylor formula and formal expansion

To prove Proposition 1.1, the main point is to replace $\mathbb{E}[f^{(|\mathbf{J}|)}(X+Y)]$ by its classical Taylor expansion of $(N - |\mathbf{J}|)^{\text{th}}$ order, so that all summand terms are of the same order and some of them can be cancelled off progressively.

Proof of Proposition 1.1 We replace $\mathbb{E}[f^{(|\mathbf{J}|)}(X+Y)]$ on the right side of (5) by

$$\sum_{k=0}^{N-|\mathbf{J}|} m_Y^{(k)}\mathbb{E}[f^{(|\mathbf{J}|+k)}(X)] + \delta_{N-|\mathbf{J}|}(f^{(|\mathbf{J}|)}, X, Y)$$

and observe that the sum of terms containing δ vanishes with $\varepsilon_N(f,X,Y)$. Hence we obtain that the right side of (5) equals

$$\sum_{d\geq 0}(-1)^d \sum_{\mathbf{J}\in\mathbb{N}_*^d,\,|\mathbf{J}|\leq N} m_Y^{(\mathbf{J})} \sum_{k=0}^{N-|\mathbf{J}|} m_Y^{(k)}\mathbb{E}[f^{(|\mathbf{J}|+k)}(X)]$$

If we split the last sum for $k=0$ and for $1\leq k\leq N-|\mathbf{J}|$ respectively, the formula above can be written as

$$\sum_{d\geq 0}(-1)^d \sum_{\mathbf{J}\in\mathbb{N}_*^d,\,|\mathbf{J}|\leq N} m_Y^{(\mathbf{J})}\mathbb{E}[f^{(|\mathbf{J}|)}(X)] + \sum_{d\geq 0}(-1)^d \sum_{\mathbf{J}\in\mathbb{N}_*^d,\,|\mathbf{J}|\leq N} m_Y^{(\mathbf{J})} \sum_{k=1}^{N-|\mathbf{J}|} m_Y^{(k)}\mathbb{E}[f^{(|\mathbf{J}|+k)}(X)]. \tag{9}$$

We make the index changes $\mathbf{J}'=(\mathbf{J},k)$ and $u=d+1$ in the second part, we find that it is just

$$\sum_{u\geq 1}(-1)^{u-1} \sum_{\mathbf{J}'\in\mathbb{N}_*^u,\,|\mathbf{J}'|\leq N} m_Y^{(\mathbf{J}')}\mathbb{E}[f^{(|\mathbf{J}'|)}(X)].$$

Thus, the terms in the first and the second parts of (9) cancel out except the one of index $d=0$ in the first part, which proves the proposition. $\square$

Using Proposition 1.1, we prove below the formal equality $\mathbb{E}[h(W)] = C_N(h) + e_N(h)$ by induction on N.

Proof of Theorem 1.2 (formal part) The equality $\mathbb{E}[h(W)] = C_0(h)+e_0(h)$ holds by definition. In the following, we assume that the equality $\mathbb{E}[h(W)] = C_k(h)+e_k(h)$ has been verified for any $k\in\{0,\cdots,N-1\}$ and for any good enough function h. By Stein's equation, $\mathbb{E}[h(W)] - C_0(h)$ is equal to

$$\sigma_W^2\mathbb{E}[f_h'(W^*) - f_h'(W)] = \sum_{i=1}^n \sigma_i^2\Big(\mathbb{E}[f_h'(W^{(i)}+X_i^*)] - \mathbb{E}[f_h'(W)]\Big).$$

Consider the following Taylor expansion

$$\mathbb{E}[f_h'(W^{(i)}+X_i^*)] = \sum_{k=0}^N m_{X_i^*}^{(k)}\mathbb{E}[f_h^{(k+1)}(W^{(i)})] + \delta_N(f_h', W^{(i)}, X_i^*).$$

By replacing $\mathbb{E}[f_h^{(k+1)}(W^{(i)})]$ in the above formula by its $(N-k)^{\text{th}}$ reverse Taylor expansion, we obtain that $\mathbb{E}[f_h'(W^{(i)}+X_i^*)]$ equals

$$\sum_{k=0}^N m_{X_i^*}^{(k)}\Bigg[\sum_{d\geq 0}(-1)^d \sum_{\substack{\mathbf{J}\in\mathbb{N}_*^d\\ |\mathbf{J}|\leq N-k}} m_{X_i}^{(\mathbf{J})}\mathbb{E}[f_h^{(|\mathbf{J}|+k+1)}(W)] + \varepsilon_{N-k}(f_h^{(k+1)}, W^{(i)}, X_i)\Bigg] + \delta_N(f_h', W^{(i)}, X_i^*).$$

Note that the term with indexes $k = d = 0$ in the sum inside the bracket is $\mathbb{E}[f_h'(W)]$. Therefore $\mathbb{E}[f_h'(W^{(i)} + X_i^*)] - \mathbb{E}[f_h'(W)]$ can be written as the sum of the following three parts

$$\sum_{k=1}^{n} m_{X_i^*}^{(k)} \sum_{d\geq 0}(-1)^d \sum_{\mathbf{J}\in\mathbb{N}_*^d,\,|\mathbf{J}|\leq N-k} m_{X_i}^{(\mathbf{J})}\mathbb{E}[f_h^{(|\mathbf{J}|+k+1)}(W)], \tag{10}$$

$$\sum_{d\geq 1}(-1)^d \sum_{\mathbf{J}\in\mathbb{N}_*^d,\,|\mathbf{J}|\leq N} m_{X_i}^{(\mathbf{J})}\mathbb{E}[f_h^{(|\mathbf{J}|+1)}(W)], \tag{11}$$

$$\sum_{k=0}^{N} m_{X_i^*}^{(k)}\varepsilon_{N-k}(f_h^{(k+1)}, W^{(i)}, X_i) + \delta_N(f_h', W^{(i)}, X_i^*). \tag{12}$$

By interchanging summations and then taking the index changes $\mathbf{K} = (\mathbf{J}, k)$ and $u = d+1$, we obtain

$$(10) = \sum_{u\geq 1}(-1)^{u-1} \sum_{\mathbf{K}\in\mathbb{N}_*^u,\,|\mathbf{K}|\leq N} m_{X_i}^{(\mathbf{K}^\circ)} m_{X_i^*}^{(\mathbf{K}^\dagger)}\mathbb{E}[f_h^{(|\mathbf{K}|+1)}(W)].$$

As the equality $m_{X_i}^{(\mathbf{J})} = m_{X_i}^{(\mathbf{J}^\circ)} m_{X_i}^{\mathbf{J}^\dagger}$ holds for any $\mathbf{J}$, (10)+(11) simplifies as

$$\sum_{d\geq 1}(-1)^{d-1} \sum_{\mathbf{J}\in\mathbb{N}_*^d,\,|\mathbf{J}|\leq N} m_{X_i}^{(\mathbf{J}^\circ)}\Big(m_{X_i^*}^{(\mathbf{J}^\dagger)} - m_{X_i}^{(\mathbf{J}^\dagger)}\Big)\mathbb{E}[f_h^{(|\mathbf{J}|+1)}(W)].$$

By the hypothesis of induction, we have $\mathbb{E}[f_h^{(|\mathbf{J}|+1)}(W)] = C_{N-|\mathbf{J}|}(f_h^{(|\mathbf{J}|+1)}) + e_{N-|\mathbf{J}|}(f_h^{(|\mathbf{J}|+1)})$, so the equality $\mathbb{E}[h(W)] = C_N(h) + e_N(h)$ follows from (7) and (8). □

3 Admissible function space

In this section, we describe the function set for which we can make the N^{th} order expansion in Theorem 1.2. We need conditions on regularity and on the increasing speed at infinity of the function h. Actually, from (7) and (8), we are concerned with the $(N-k)^{\text{th}}$ order expansion of $f_h^{(k+1)}$ for $k = 1, \cdots, N$. So it would be natural to expect that f_h' still belongs to this set. Then by a recursive procedure, all terms will be well defined.

Recall ([11], Chapter VI) that any function g on $\mathbb{R}$ which is locally of finite variation can be uniquely decomposed into the sum of a function of pure jump and a continuous function locally of finite variation and vanishing at the origin. That is, $g = g_c + g_d$ where g_c is called the *continuous part* of g and g_d is the *purely discontinuous part*.

Let $\alpha \in (0, 1]$ and $p \geq 0$ be two real numbers. For any function f on $\mathbb{R}$, the following quantity has been defined by Barbour in [1]:

$$\|f\|_{\alpha,p} := \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha(1 + |x|^p + |y|^p)}. \tag{13}$$

The finiteness of this quantity implies that the function f is locally α-Lipschitz, and the increasing speed of f at infinity is at most of order $|x|^{\alpha+p}$. All functions f such that $\|f\|_{\alpha,p} < +\infty$ forms a vector space over $\mathbb{R}$, and $\|\cdot\|_{\alpha,p}$ is a norm on it. We list below several properties of $\|\cdot\|_{\alpha,p}$, which will be useful afterwards and we leave the proofs in the Appendix A.

Lemma 3.1 *Let f be a function on $\mathbb{R}$, $\alpha,\beta \in (0,1]$ and $p,q \geq 0$.*

1) If $p \leq q$, then $\|f\|_{\alpha,p} < +\infty$ implies $\|f\|_{\alpha,q} < +\infty$.

2) If $\alpha \leq \beta$, then $\|f\|_{\beta,p} < +\infty$ implies $\|f\|_{\alpha,p+\beta-\alpha} < +\infty$.

3) If P is a polynomial of degree d, then $\|f\|_{\alpha,p} < +\infty$ implies $\|Pf\|_{\alpha,p+d} < +\infty$.

4) Assume that F is a primitive function of f, then $\|f\|_{\alpha,p} < +\infty$ implies $\|F\|_{1,p+\alpha} < +\infty$. (Hence $\|F\|_{\alpha,p+1} < +\infty$ by 2).)

Inspired by [1], we introduce the following function space.

Definition 3.2 Let $N \geq 0$ be an integer, and $\alpha \in (0,1]$, $p \geq 0$ be two real numbers. Denote by $\mathcal{H}^N_{\alpha,p}$ the vector space of all Borel functions h on $\mathbb{R}$ verifying the following conditions:

a) h has N^{th} order derivative which is locally of finite variation and which has finitely many jumps,

b) the continuous part of $h^{(N)}$ satisfies $\|h_c^{(N)}\|_{\alpha,p} < +\infty$.

Condition a) implies that the pure jump part of $h^{(N)}$ is bounded. Condition b) implies that $h_c^{(N)}$ has at most polynomial increasing speed at infinity, therefore also is h. These conditions allow us to include some irregular functions such as indicator functions. Let k be a real number and $I_k(x) = 1\!\!1_{\{x\leq k\}}$. Then $\|I_k\|_{\alpha,p}$ is clearly not finite. However, $\|I_{k,c}\|_{\alpha,p} = 0$, which means that for any $\alpha \in (0,1]$ and any $p \geq 0$, $I_k(x) \in \mathcal{H}^0_{\alpha,p}$. Note that any function h in $\mathcal{H}^0_{\alpha,p}$ can be decomposed as $h = h_c + h_d$, where h_c satisfies $\|h_c\|_{\alpha,p} < +\infty$, the discontinuous part h_d is a linear combination of indicator functions of the form $1\!\!1_{\{x\leq k\}}$ plus a constant (so that $h_c(0) = 0$).

Proposition 3.3 *Let $N \geq 0$ be an integer, $\alpha,\beta \in (0,1]$ and $p,q \geq 0$ be real numbers. Then the following assertions hold:*

1) when $N \geq 1$, $h \in \mathcal{H}^N_{\alpha,p}$ if and only if $h' \in \mathcal{H}^{N-1}_{\alpha,p}$;

2) if $p \leq q$, then $\mathcal{H}^N_{\alpha,p} \subset \mathcal{H}^N_{\alpha,q}$; if $\alpha \leq \beta$, then $\mathcal{H}^N_{\beta,p} \subset \mathcal{H}^N_{\alpha,p+\beta-\alpha}$;

3) when $N \geq 1$, $\mathcal{H}^N_{\alpha,p} \subset \mathcal{H}^{N-1}_{1,\alpha+p} \subset \mathcal{H}^{N-1}_{\alpha,p+1}$;

4) if $h \in \mathcal{H}^N_{\alpha,p}$ and if P is a polynomial of degree d, then $Ph \in \mathcal{H}^N_{\alpha,p+d}$.

Proof. 1) results from the definition. 2), 3) and 4) are consequences of Lemma 3.1. □

The following result on the operator $h \to f_h$ is fundamental. It shows that compared to h, the solution of Stein's equation f_h has one more order in regularity and its derivative has the same order in increasing speed at infinity. The proof of this proposition, which is rather technical, is postponed to Appendix B.

Proposition 3.4 *Assume that $h \in \mathcal{H}^N_{\alpha,p}$. Then $f_h \in \mathcal{H}^{N+1}_{\alpha,p}$.*

We now restate Theorem 1.2 in the function space context.

Theorem 3.5 *Let $N \geq 0$ be an integer, $\alpha \in (0,1]$ and $p \geq 0$. Assume that $h \in \mathcal{H}^N_{\alpha,p}$. Let $X_1, \cdots, X_n$ be zero-mean random variables which have $(N+\max(\alpha+p,2))^{\text{th}}$ order moment. Then all terms in (7) and (8) are well defined, and the equality $\mathbb{E}[h(W)] = C_N(h) + e_N(h)$ holds.*

Proof. When $N = 0$, $h \in \mathcal{H}^0_{\alpha,p}$ and then $h(x) = O(|x|^{\alpha+p})$. Hence $\mathbb{E}[h(W)]$ and $\Phi_{\sigma_W}(h)$ are well defined. Assume that we have proved the theorem for $0, \cdots, N-1$. Let $h \in \mathcal{H}^N_{\alpha,p}$. Then by Proposition 3.3, $h(x) \in \mathcal{H}^N_{\alpha,p} \subset \mathcal{H}^{N-1}_{\alpha,p+1} \cdots \subset \mathcal{H}^0_{\alpha,p+N}$, so $h(x) = O(|x|^{\alpha+p+N})$. By Proposition 3.4, $f_h \in \mathcal{H}^{N+1}_{\alpha,p}$ and by Proposition 3.3 1), for any $|\mathbf{J}| = 1, \cdots, N$, $f_h^{(|\mathbf{J}|+1)} \in \mathcal{H}^{N-|\mathbf{J}|}_{\alpha,p}$. So the induction hypothesis implies that $C_{N-|\mathbf{J}|}(f_h^{(|\mathbf{J}|+1)})$ and $e_{N-|\mathbf{J}|}(f_h^{(|\mathbf{J}|+1)})$ exist. Furthermore, for the terms ε_{N-k} and δ_N in (8), since $f_h^{(k+1)}(x) = O(|x|^{\alpha+p+N-k})$ for any $k = 0, \cdots, N$, they are well defined. Finally, combined with the equality

$$\mathbb{E}[(X_i^*)^k] = \frac{\mathbb{E}[X_i^{k+2}]}{\sigma_i^2(k+1)},$$

all moments figuring in (7) and (8) exist. Thus all terms are well defined, and the formal proof in the previous section shows that $\mathbb{E}[h(W)] = C_N(h) + e_N(h)$. □

4 Error estimations

4.1 Concentration inequalities

We shall prove some concentration inequalities similar to several results in [3, 4], which give upper bounds for probabilities of the form $\mathbb{P}(a \leqslant W \leqslant b)$ with a and b being two real numbers. We shall take into consideration the parameter α and give some variants where appear certain lower order moments if $\alpha < 1$. When $\alpha = 1$, we recover some estimations in [5]. These concentration inequalities will be useful to estimate the approximation error terms and the proof is based on the zero bias transformation.

Lemma 4.1 *Let $\alpha \in (0,1]$ be a real number and X be a r.v. with mean zero, finite variance $\sigma^2 > 0$ and up to $(\alpha+2)^{\text{th}}$ order moments. Let X^* have the zero biased distribution of X and be independent of X. Then, for any $\varepsilon > 0$,*

$$\mathbb{P}(|X - X^*| > \varepsilon) \leq \frac{1}{2\varepsilon^\alpha(\alpha+1)\sigma^2}\mathbb{E}[|X^s|^{\alpha+2}],$$

where $X^s = X - \widetilde{X}$ and $\widetilde{X}$ is an independent copy of X.

Proof. Similar to the Markov inequality, the following inequality holds:

$$\mathbb{P}(|X - X^*| > \varepsilon) \leq \frac{1}{\varepsilon^\alpha}\mathbb{E}[|X - X^*|^\alpha].$$

Moreover, since X and X^* are independent, the definition of the zero bias transformation (see [5, Pro2.3]) implies that

$$\mathbb{E}[|X - X^*|^\alpha] = \frac{1}{2(\alpha+1)\sigma^2}\mathbb{E}[|X^s|^{\alpha+2}].$$

□

Proposition 4.2 *Let X_i $(i = 1, \cdots, n)$ be independent random variables with mean zero and variance $\sigma_i^2 > 0$. Let $W = X_1 + \cdots + X_n$ and denote its variance by σ_W^2. For $a, b \in \mathbb{R}$ such that $a \leq b$ and any real number $\alpha \in (0,1]$, we have*

$$\mathbb{P}(a \leq W \leq b) \leq 2\Big(\frac{b-a}{2\sigma_W}\Big)^\alpha + \frac{2}{\alpha+1}\sum_{i=1}^n \mathbb{E}\Big[\big|\frac{X_i^s}{\sigma_W}\big|^{\alpha+2}\Big] + \frac{1}{2\sigma_W^2}\Big(\sum_{i=1}^n \sigma_i^4\Big)^{\frac{1}{2}}. \tag{14}$$

Proof. Let $I_{[a,b]}(x) = 1$ if $x \in [a,b]$ and $I_{[a,b]}(x) = 0$ otherwise. Its primitive function $f(x) := \int_{(a+b)/2}^x I_{[a,b]}(t)dt$ satisfies $|f(x)| \leq (b-a)/2$. Then

$$\mathbb{E}(I_{[a,b]}(W^*)) = \frac{1}{\sigma_W^2}\mathbb{E}(Wf(W)) \leq \min\left(\frac{b-a}{2\sigma_W}, 1\right).$$

Note that for any $u \geq 0$ and any $\alpha \in (0,1]$, $\min(u,1) \leq u^\alpha$. Then for any $\varepsilon > 0$,

$$\mathbb{P}(a - \varepsilon \leq W^* \leq b + \varepsilon) \leq \left(\frac{b-a+2\varepsilon}{2\sigma_W}\right)^\alpha \leq \left(\frac{b-a}{2\sigma_W}\right)^\alpha + \left(\frac{\varepsilon}{\sigma_W}\right)^\alpha$$

where the last inequality is because for any u and v positive, one always has $(u+v)^\alpha \leq u^\alpha + v^\alpha$. On the other hand, by using a conditional expectation technique,

$$\begin{aligned}\mathbb{P}(a - \varepsilon \leq W^* \leq b + \varepsilon) &\geq \mathbb{P}(a \leq W \leq b, |X_I - X_I^*| \leq \varepsilon) \\ &\geq \mathbb{P}(a \leq W \leq b)\mathbb{P}(|X_I^* - X_I| \leq \varepsilon) - \frac{1}{4}\Big(\sum_{i=1}^n \frac{\sigma_i^4}{\sigma_W^4}\Big)^{\frac{1}{2}}.\end{aligned}$$

We recall that $W^* = W^{(I)} + X_I^*$ where I is a random variable taking values in $\{1, \cdots, n\}$ with $\mathbb{P}(I = i) = \sigma_i^2/\sigma_W^2$, $W^{(i)} := W - X_i$, X_i^* has the zero biased distribution of X_i and is independent of $W^{(i)}$. In this proof exceptionally, we assume that X_i^* is also independent of X_i. By Lemma 4.1,

$$\mathbb{P}(|X_I^* - X_I| \leq \varepsilon) = 1 - \sum_{i=1}^n \frac{\sigma_i^2}{\sigma_W^2} \mathbb{P}(|X_i^* - X_i| > \varepsilon) \geq 1 - \frac{1}{2\sigma_W^2(\alpha+1)\varepsilon^\alpha} \sum_{i=1}^n \mathbb{E}[|X_i^s|^{\alpha+2}].$$

Finally, the inequality (14) follows by taking

$$\varepsilon = \Big(\frac{1}{\sigma_W^2(\alpha+1)} \sum_{i=1}^n \mathbb{E}[|X_i^s|^{\alpha+2}] \Big)^{\frac{1}{\alpha}}.$$

□

Corollary 4.3 *Let $a, b \in \mathbb{R}$ such that $a \leq b$ and $i \in \{1, \cdots, n\}$, then*

$$\mathbb{P}(a \leq W^{(i)} \leq b) \leq 4\Big(\frac{b-a}{2\sigma_W}\Big)^\alpha + \frac{4}{\alpha+1} \sum_{j=1}^n \mathbb{E}\Big[\big| \frac{X_j^s}{\sigma_W} \big|^{\alpha+2} \Big] + \frac{1}{\sigma_W^2} \big(\sum_{j=1}^n {\sigma_j}^4 \big)^{\frac{1}{2}} + 4\Big(\frac{2\sigma_i}{\sigma_W}\Big)^\alpha$$

where $W^{(i)} = W - X_i$.

Proof. Let $\varepsilon > 0$ be a real number, then

$$\mathbb{P}(a \leq W^{(i)} \leq b, \ |X_i| \leq \varepsilon) \leq \mathbb{P}(a - \varepsilon \leq W \leq b + \varepsilon).$$

Note that $W^{(i)}$ and X_i are independent and

$$\mathbb{P}(|X_i| \leq \varepsilon) = 1 - \mathbb{P}(|X_i| > \varepsilon) \geq 1 - \frac{E[|X_i|]}{\varepsilon}.$$

By Proposition 4.2 and taking $\varepsilon = 2\mathbb{E}[|X_i|]$, we obtain the inequality. □

4.2 Estimations of error terms

In this section, we shall estimate the error term $e_N(h)$ in Theorem 3.5. The recursive formulas (8) and (6) permit us to reduce the problem to the estimation of classical Taylor expansion errors.

For any positive random variable Y and any real number $\beta \geq 0$, we introduce the notation $m_Y^{(\beta)} := \mathbb{E}[Y^\beta]/\Gamma(\beta+1)$, where Γ is the Gamma function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. This notation generalizes the one introduced in §1 since when $\beta \in \mathbb{N}$, $\Gamma(\beta+1) = \beta!$.

Proposition 4.4 *Let $N \geq 0$ be an integer, $\alpha \in (0,1]$ and $p \geq 0$ be two real numbers. Let X be a random variable which has up to $(N+\alpha+p)^{\text{th}}$ moments and satisfies the following concentration inequality*

$$\mathbb{P}(a \leq X \leq b) \leq c(b-a)^\alpha + r, \qquad \forall a, b \in \mathbb{R},\, a \leq b,$$

where c and r are two constants. Let Y be a random variable which is independent of X and has up to $(N+\alpha+p)^{\text{th}}$ moments. Then, for any function $g \in \mathcal{H}^N_{\alpha,p}$ and any $k = 0, \cdots, N$,

$$\begin{aligned}\big|\delta_{N-k}(g^{(k)}, X, Y)\big| \leq\ & V(g_d^{(N)})\Big(cm_{|Y|}^{(N-k+\alpha)} + rm_{|Y|}^{(N-k)}\Big) \\ &+ \|g_c^{(N)}\|_{\alpha,p}\Big(u_{\alpha,p,X}m_{|Y|}^{(N-k+\alpha)} + v_{\alpha,p}m_{|Y|}^{(N-k+\alpha+p)}\Big),\end{aligned} \tag{15}$$

where $V(g_d^{(N)})$ denotes the total variation of $g_d^{(N)}$, the coefficients $u_{\alpha,p,X}$ and $v_{\alpha,p}$ are defined as $u_{\alpha,p,X} = \big(1 + (1+2^p)\mathbb{E}[|X|^p]\big)\Gamma(\alpha+1)$ and $v_{\alpha,p} = 2^p\Gamma(\alpha+p+1)$.

Proof. We have by (4) that when $k < N$,

$$\delta_{N-k}(g^{(k)}, X, Y) = \frac{1}{(N-k-1)!}\int_0^1 (1-t)^{N-k-1}\mathbb{E}\big[\big(g^{(N)}(X+tY) - g^{(N)}(X)\big)Y^{N-k}\big]\,dt.$$

Since $g \in \mathcal{H}^N_{\alpha,p}$, the function $g_d^{(N)}$ can be written as

$$g_d^{(N)}(x) = g_d^{(N)}(0) + \sum_{1\leq j\leq M} \varepsilon_j \mathbb{1}_{x\leq K_j} - \sum_{\substack{1\leq j\leq M\\ K_j\geq 0}} \varepsilon_j.$$

Therefore, $g_d^{(N)}(X+tY) - g_d^{(N)}(X) = \sum_{j=1}^M \varepsilon_j \mathbb{1}_{K_j - tY_+ < X \leq K_j - tY_-}$, where $Y_+ = \max(Y,0)$ and $Y_- = \min(Y,0)$. Thus the concentration inequality hypothesis implies that

$$\mathbb{E}\big[|g_d^{(N)}(X+tY) - g_d^{(N)}(X)|\ \big|Y\big] \leq \sum_{j=1}^M |\varepsilon_j|\big(ct^\alpha|Y|^\alpha + r\big).$$

Moreover, one has

$$\int_0^1 \frac{(1-t)^{N-k-1}}{(N-k-1)!}\mathbb{E}\Big[\sum_{j=1}^M |\varepsilon_j|\big(ct^\alpha|Y|^\alpha + r\big)|Y|^{N-k}\Big]dt = \sum_{j=1}^M |\varepsilon_j|\big(cm_{|Y|}^{(N-k+\alpha)} + rm_{|Y|}^{(N-k)}\big)$$

by using the following equality concerning Beta function

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0.$$

On the other hand, by definition of the norm $\|\cdot\|_{\alpha,p}$, we have

$$\begin{aligned}\big|g_c^{(N)}(X+tY) - g_c^{(N)}(X)\big| &\leq \|g_c^{(N)}\|_{\alpha,p}|tY|^\alpha(1 + |X+tY|^p + |X|^p) \\ &\leq \|g_c^{(N)}\|_{\alpha,p}|tY|^\alpha\big(1 + (2^p+1)|X|^p + 2^p|tY|^p\big),\end{aligned}$$

where the last inequality results from $(a+b)^p \leq 2^p(a^p+b^p)$. Note that

$$\int_0^1 \frac{(1-t)^{N-k-1}}{(N-k-1)!}\mathbb{E}\Big[|tY|^\alpha\big(1+(2^p+1)|X|^p+2^p|tY|^p\big)|Y|^{N-k}\Big]dt = u_{\alpha,p,X}m_{|Y|}^{(N-k+\alpha)}+v_{\alpha,p}m_{|Y|}^{(N-k+\alpha+p)}.$$

Thus we obtain the estimation (15).
Finally, it remains to check the case when $k=N$. Consider the continuous and discontinuous parts of $\delta_0(g^{(N)},X,Y)=\mathbb{E}\big[g^{(N)}(X+Y)-g^{(N)}(X)\big]$ respectively. By using similar method as above, we obtain that $\mathbb{E}\big[|g_d^{(N)}(X+Y)-g_d^{(N)}(X)|\big] \leq V(g_d^{(N)})\big(cm_{|Y|}^{(1)}+r\big)$ and $\mathbb{E}\big[|g_c^{(N)}(X+Y)-g_c^{(N)}(X)|\big] \leq \|g_c^{(N)}\|_{\alpha,p}\left(\mathbb{E}[|Y|^\alpha]\big(1+(2^p+1)\mathbb{E}[|X|^p]\big)+2^p\mathbb{E}[|Y|^{\alpha+p}]\right)$, which implies (15). □

By Proposition 1.1 and Proposition 4.4, we obtain the error estimation for the reverse Taylor expansion.

Corollary 4.5 *With the previous notation, we have*

$$\begin{aligned}|\varepsilon_N(g,X,Y)| \leq \sum_{d\geq 0}\sum_{\mathbf{J}\in\mathbb{N}_*^d,\,|\mathbf{J}|\leq N} m_{|Y|}^{(\mathbf{J})}\Big[&V(g_d^{(N)})\big(cm_{|Y|}^{(N-|\mathbf{J}|+\alpha)}+rm_{|Y|}^{(N-|\mathbf{J}|)}\big)\\ &+\|g_c^{(N)}\|_{\alpha,p}\big(u_{\alpha,p,X}m_{|Y|}^{(N-|\mathbf{J}|+\alpha)}+v_{\alpha,p}m_{|Y|}^{(N-|\mathbf{J}|+\alpha+p)}\big)\Big],\end{aligned} \tag{16}$$

Combining the concentration inequality (Corollary 4.3) and the above estimations (Proposition 4.4 and Corollary 4.5), we obtain upper bounds for the Taylor and reverse Taylor remainders $\delta_{N-k}(f_h^{(k)},W^{(i)},X_i)$ and $\varepsilon_{N-k}(f_h^{(k)},W^{(i)},X_i)$, where the summand variables $X_1,\cdots,X_n$ are independent. This allows us, together with the recursive formula (8), to obtain an upper bound for the asymptotic expansion remainder $e_N(h)$.

In particular, we give in the following the order estimation of $e_N(h)$ when $X_1,\cdots,X_n$ are in addition i.i.d. random variables.

Proposition 4.6 *Suppose that $X_1,\cdots,X_n$ are i.i.d. random variables with mean zero and up to $(N+2+\alpha+p)^{th}$ order moments, normalized such that each X_i has the same distribution as $Z/\sqrt{n}$ where Z is a fixed random variable with mean zero and finite non-zero variance. Then for any function $g\in\mathcal{H}_{\alpha,p}^N$ and any $k=0,\cdots,N$, we have*

$$\delta_{N-k}(g^{(k)},W^{(i)},X_i^*) = O\left(\Big(\frac{1}{\sqrt{n}}\Big)^{N-k+\alpha+p}\right), \tag{17}$$

$$\varepsilon_{N-k}(g^{(k)},W^{(i)},X_i) = O\left(\Big(\frac{1}{\sqrt{n}}\Big)^{N-k+\alpha+p}\right), \tag{18}$$

where $W^{(i)}=W-X_i$ and X_i^* is independent of $W^{(i)}$. The implied constants depend on $\|g_c^{(N)}\|_{\alpha,p}$, $V(g_d^{(N)})$ and up to $(N-k+2+\alpha+p)^{th}$ order moments of Z.

Proof. By Corollary 4.3, we have for any $a \leq b$ and any $\alpha \in (0,1]$ that

$$\mathbb{P}(a \leq W^{(i)} \leq b) \leq c(b-a)^\alpha + r(n)$$

where the coefficients are given by

$$c = \frac{2^{2-\alpha}}{\sigma^\alpha}, \qquad r(n) = \frac{4}{\sigma^{2+\alpha}(\alpha+1)} \frac{\mathbb{E}[|Z^s|^{\alpha+2}]}{\sqrt{n}^\alpha} + \frac{1}{\sqrt{n}} + \frac{8}{\sqrt{n}^\alpha}.$$

By Proposition 4.4, we obtain an upper bound of $\delta_{N-k}(g^{(k)}, W^{(i)}, X_i^*)$ which is determined by a linear combination of terms (with coefficient not depending on n):

$$m_{|X_i^*|}^{(N-k+\alpha)}, \ r(n) m_{|X_i^*|}^{(N-k)}, \ \mathbb{E}[|W^{(i)}|^p] m_{|X_i^*|}^{(N-k+\alpha)} \ \text{ and } \ m_{|X_i^*|}^{(N-k+\alpha+p)}. \tag{19}$$

Note that $r(n) = O((1/\sqrt{n})^\alpha)$. For any $k = 0, \cdots, N$, $\mathbb{E}[|X_i^*|^k]$ equals $\mathbb{E}[|X_i|^{k+2}]/(\sigma_i^2(k+1))$ and is of order $(1/\sqrt{n})^k$. So the first three terms in (19) are of order $(1/\sqrt{n})^{N-k+\alpha}$ and the last term is of order $(1/\sqrt{n})^{N-k+\alpha+p}$, which implies the first assertion. The second assertion then follows by Corollary 4.5.

□

Remark 4.7 According to (15) and (16), the implicit constants in (17) and (18) can be explicitly calculated.

Proposition 4.8 *Let $N \geq 0$ be an integer, $\alpha \in (0,1]$ and $p \geq 0$ be two real numbers. Let h be a function in $\mathcal{H}_{\alpha,p}^N$, and $X_1, \cdots, X_n$ be as in Proposition 4.6. Then the error term $e_N(h)$ defined in* (8) *satisfies*

$$e_N(h) = O\left(\left(\frac{1}{\sqrt{n}}\right)^{N+\alpha+p}\right),$$

where the implied constant depends on up to $(N+2+\alpha+p)^{\text{th}}$ order moment of Z.

Proof. We prove the theorem by induction on N. When $N = 0$,

$$e_0(h) = \sum_{i=1}^n \sigma_i^2 \Big(\delta_0(f_h', W^{(i)}, X_i^*) + \varepsilon_0(f_h', W^{(i)}, X_i) \Big).$$

Since $h \in \mathcal{H}_{\alpha,p}^0$, $f_h \in \mathcal{H}_{\alpha,p}^1$. Then by Proposition 4.6, $e_0(h) = O((1/\sqrt{n})^{\alpha+p})$.

Assume that we have already proved the theorem for $0, \cdots, N-1$. Consider $h \in \mathcal{H}_{\alpha,p}^N$ and $e_N(h)$ defined as in (8). For any $\mathbf{J}$ such that $1 \leq |\mathbf{J}| \leq N$, $e_{N-|\mathbf{J}|} = O((1/\sqrt{n})^{N-|\mathbf{J}|+\alpha+p})$. In addition, since $|\mathbf{J}^\circ| + |\mathbf{J}^\dagger| = |\mathbf{J}|$, we have that $m_{X_i}^{(\mathbf{J}^\circ)} \big(m_{X_i^*}^{(\mathbf{J}^\dagger)} - m_{X_i}^{(\mathbf{J}^\dagger)} \big)$ is of order $(1/\sqrt{n})^{|\mathbf{J}|}$. On the other hand, $f_h' \in \mathcal{H}_{\alpha,p}^N$, so $\delta_N(f_h', W^{(i)}, X_i^*) = O((1/\sqrt{n})^{N+\alpha+p})$. Moreover, for any $k = 0, \cdots, N$, $f_h^{(k+1)} \in \mathcal{H}_{\alpha,p}^{N-k}$. So $\varepsilon_{N-k}(f_h^{(k+1)}, W^{(i)}, X_i^*) = O((1/\sqrt{n})^{N-k+\alpha+p})$. Finally we have $m_{X_i^*}^{(k)} = O((1/\sqrt{n})^k)$. Combining all the above estimations, we prove the proposition. □

Consider now several examples. Let $I_k(x) = \mathbb{1}_{\{x\le k\}}$ be the indicator function. As mentioned before, $I_k \in \mathcal{H}^0_{\alpha,0}$. By Proposition 4.8, we know that if $X_1, \cdots, X_n$ are i.i.d. random variables with up to $(2+\alpha)^{\text{th}}$ order moment, then $e_0(h) = O((1/\sqrt{n})^\alpha)$, where the coefficient depends on up to $(2+\alpha)^{\text{th}}$ moment of the summand variables. This is similar to a result (Theorem 6) in [13, §V.3]. When $\alpha = 1$, it corresponds to the order estimation in the classical Berry-Esseen inequality.

Let $h(x) = (x-k)_+$ be the call function discussed in [5]. As a primitive function of the indicator function, we know that $h \in \mathcal{H}^1_{\alpha,0}$. So the call function admits a first order expansion given by (7) as:

$$C_1(h) = \Phi_{\sigma_W}(h) + \sum_{i=1}^n \sigma_i^2 \mathbb{E}[X_i^*]\Phi_{\sigma_W}(f_h'').$$

Moreover, since $\sigma_W^2\Phi_{\sigma_W}(f_h'') = \Phi_{\sigma_W}(xf_h') = \frac{1}{\sigma_W^2}\Phi_{\sigma_W}\big((\frac{x^2}{3\sigma_W^2}-1)xh(x)\big)$. We recover the correction term in [5].

A Proof of Lemma 3.1

Proof. For the first two assertions, it suffices to prove respectively the boundness of the following two functions

$$\frac{1+|x|^p+|y|^p}{1+|x|^q+|y|^q}, \qquad |x-y|^{\beta-\alpha}\frac{1+|x|^p+|y|^p}{1+|x|^{p+\beta-\alpha}+|y|^{p+\beta-\alpha}}.$$

These functions are both continuous on $\mathbb{R}$, therefore are bounded on any compact subset of $\mathbb{R}^2$. Thus we may assume without loss of generality that $r = \sqrt{x^2+y^2} \ge 1$. In this case, $\max\{|x|,|y|\} \ge r/\sqrt{2}$, so

$$\frac{1+|x|^p+|y|^p}{1+|x|^q+|y|^q} \le \frac{1+2r^p}{1+(r/\sqrt{2})^q} \le 3\cdot 2^{q/2},$$

$$|x-y|^{\beta-\alpha}\frac{1+|x|^p+|y|^p}{1+|x|^{p+\beta-\alpha}+|y|^{p+\beta-\alpha}} \le (2r)^{\beta-\alpha}\frac{1+2r^p}{1+(r/\sqrt{2})^{p+\beta-\alpha}} \le 3\cdot 2^{(p+3\beta-3\alpha)/2}.$$

3) One has

$$\frac{|P(x)f(x)-P(y)f(y)|}{|x-y|^\alpha(1+|x|^{p+d}+|y|^{p+d})} \le \frac{(1+|x|^p+|y|^p)P(x)}{1+|x|^{p+d}+|y|^{p+d}}\|f\|_{\alpha,p} + \frac{|f(y)|\cdot|P(x)-P(y)|}{|x-y|^\alpha(1+|x|^{p+d}+|y|^{p+d})}.$$

By using the argument as in the proof of 1) and 2), we obtain that the first term in the right-hand side is bounded. Since P is a polynomial of degree d, there exists a polynomial $Q(x,y)$ in two variables and of degree $d-1$, such that $Q(x,y) = (P(x)-P(y))/(x-y)$. Therefore, the second term equals

$$\frac{|Q(x,y)|\cdot|x-y|^{1-\alpha}\cdot|f(y)|}{1+|x|^{p+d}+|y|^{p+d}}$$

which is bounded by a similar argument as for proving 1) and 2).

4) Since $\|f\|_{\alpha,p} < +\infty$, $|f(t)| \ll 1 + |t|^{\alpha+p}$. Therefore, for $x, y \in \mathbb{R}$, $x \le y$, one has

$$|F(x) - F(y)| \le \int_x^y |f(t)|\,dt \ll \int_x^y \left(1 + |t|^{\alpha+p}\right) dt \le (1 + |x|^{\alpha+p} + |y|^{\alpha+p})|y - x|.$$

Hence $\dfrac{|F(x) - F(y)|}{|x - y|(1 + |x|^{p+\alpha} + |y|^{p+\alpha})}$ is bounded. □

B Proof of Proposition 3.4

We now prove the Proposition 3.4. Let $h \in \mathcal{H}^N_{\alpha,p}$. The function f_h is one more order differentiable than h and is hence $N+1$ times differentiable. Taking N^{th} order derivative on both sides of Stein's equation, we get

$$(xf_h(x))^{(N)} - \sigma^2 f_h^{(N+1)}(x) = h^{(N)}(x). \tag{20}$$

The function $(xf_h(x))^{(N)}$ is continuous, so $f_h^{(N+1)}$ is locally of finite variation and has finitely many jumps as $h^{(N)}(x)$ does. In the following, we shall prove $\|f_{h,c}^{(N+1)}\|_{\alpha,p} < +\infty$.

Definition B.1 Let A be an interval in $\mathbb{R}$ and f be a Borel function on A. For any $\alpha \in (0, 1]$ and $p \ge 0$, we define

$$\|f\|^A_{\alpha,p} := \sup_{\substack{x \neq y \\ x,y \in A}} \frac{|f(x) - f(y)|}{|x - y|^\alpha (1 + |x|^p + |y|^p)}. \tag{21}$$

This definition is analogous to (13), restricted to an interval. When A avoids an open neighborhood of 0, then the finiteness of $\|f\|^A_{\alpha,p}$ is equivalent to that of $\sup_{\substack{x \neq y \\ x,y \in A}} \frac{|f(x)-f(y)|}{|x-y|^\alpha(|x|^p+|y|^p)}$. This property does not hold for the norm $\|.\|_{\alpha,p}$ defined in (13). As a consequence, we have the following result.

Lemma B.2 *Let $A \subset (-\infty, -1] \cup [1, +\infty)$ be an interval, $\alpha \in (0, 1]$ and $p \geqslant 0$. Let q be a real number such that $0 \leqslant q \leqslant p$. Then for any Borel function f defined on A, $\|f\|^A_{\alpha,p} < +\infty$ if and only if $\|f(x)/x^{p-q}\|^A_{\alpha,q} < +\infty$.*

Proof. If $\|f(x)/x^{p-q}\|^A_{\alpha,q} < +\infty$, then by similar arguments as for proving Lemma 3.1, we have $\|f\|^A_{\alpha,p} < +\infty$. We now consider the converse assertion. Firstly, there exists a constant $C > 0$ such that $|f(x)| \leqslant C|x|^{\alpha+p}$ for any $x \in A$. For any $x, y \in A$, $|x| < |y|$,

$$\frac{|f(x)x^{q-p} - f(y)y^{q-p}|}{|x - y|^\alpha(1 + |x|^q + |y|^q)} \leqslant |f(x)| \frac{|x^{q-p} - y^{q-p}|}{|x - y|^\alpha(1 + |x|^q + |y|^q)} + |y|^{q-p} \frac{|f(x) - f(y)|}{|x - y|^\alpha(1 + |x|^q + |y|^q)}.$$

The second term is finite since

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha(|y|^{p-q}+|x|^q|y|^{p-q}+|y|^p)} \le \frac{|f(x)-f(y)|}{|x-y|^\alpha(1+|x|^p+|y|^p)} = \|f\|_{\alpha,p}^A.$$

By the mean value theorem, the first term is bounded by

$$C|x|^{\alpha+p}\frac{|x-y|\cdot|q-p|\cdot|x|^{q-p-1}}{|x-y|^\alpha(1+|x|^q+|y|^q)}$$

and thus by $C|q-p|$ if we assume in addition that $|y|<2|x|$. When $|y|\geqslant 2|x|$, one has

$$|f(x)|\frac{|x^{q-p}-y^{q-p}|}{|x-y|^\alpha(1+|x|^q+|y|^q)} \leqslant C\frac{|x|^{\alpha+p}|x|^{q-p}}{|x|^{\alpha+q}} \leqslant C.$$

□

The following lemma allows us to consider the estimations on several disjoint intervals respectively.

Lemma B.3 *If $A=A_1\cup A_2$ where A_1 and A_2 are two intervals such that $A_1\cap A_2\neq\emptyset$, then*

$$\sup\{\|f\|_{\alpha,p}^{A_1},\|f\|_{\alpha,p}^{A_2}\} \le \|f\|_{\alpha,p}^A \le 2(\|f\|_{\alpha,p}^{A_1}+\|f\|_{\alpha,p}^{A_2}).$$

Proof. The first inequality is obvious. For the second inequality, we only need to prove for any $x\in A_1$ and $y\in A_2$ that

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha(1+|x|^p+|y|^p)} \le 2(\|f\|_{\alpha,p}^{A_1}+\|f\|_{\alpha,p}^{A_2}).$$

Without loss of generality, we may suppose that $A_1\cap A_2$ contains a single point z. Then $|f(x)-f(y)|\le|f(x)-f(z)|+|f(y)-f(z)|$. In addition, since z is between x and y, we have $|x-y|\ge\max(|x-z|,|y-z|)$ and $|x|^p+|y|^p\ge\frac{1}{2}\max(|x|^p+|z|^p,|y|^p+|z|^p)$. So

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha(1+|x|^p+|y|^p)} \le 2\left(\frac{|f(x)-f(z)|}{|x-z|^\alpha(1+|x|^p+|z|^p)}+\frac{|f(z)-f(y)|}{|z-y|^\alpha(1+|z|^p+|y|^p)}\right),$$

which implies the second inequality. □

Lemma B.4 *If $h\in\mathcal{H}_{\alpha,p}^N$, then $\|f_{h,c}^{(N+1)}\|_{\alpha,p}^A<+\infty$ for any bounded interval A.*

Proof. Firstly, for any bounded interval A and any Borel function g, $\|g\|_{\alpha,p}^A<+\infty$ if and only if g is α-Lipschitz on A. We examine $f_{h,c}^{(N+1)}$ using (20). Since $h\in\mathcal{H}_{\alpha,p}^N$, $h_c^{(N)}$ is locally α-Lipschitz. The function $(xf_h(x))^{(N+1)}=xf_h^{(N+1)}(x)+(N+1)f_h^{(N)}(x)$ has finitely many jumps. Hence $(xf_h(x))^{(N)}$ is a primitive function of a locally bounded function, thus is locally 1-Lipschitz. So by (20), $f_{h,c}^{(N+1)}$ is locally α-Lipschitz, which implies the lemma. □

Let $A_1 = [-1, 1]$, $A_2 = (-\infty, -1]$ and $A_3 = [1, +\infty)$. Lemma B.3 shows that to prove the finiteness of $\|f_{h,c}^{(N+1)}\|$, it suffices to prove respectively the finiteness of $\|f_{h,c}^{(N+1)}\|_{\alpha,p}^{A_i}$, ($i = 1, 2, 3$). Lemma B.4 shows that $\|f_{h,c}^{(N+1)}\|_{\alpha,p}^{[-1,1]} < +\infty$. So it remains to deal with $f_{h,c}^{(N+1)}$ on the set $A_2 \cup A_3 = \mathbb{R} \setminus (-1, 1)$. To this end, we introduce a "modified" Stein's equation as in [5, Appendix]:

$$x\widetilde{f}_h(x) - \sigma^2 \widetilde{f}_h'(x) = h(x), \quad x \in \mathbb{R} \setminus (-1, 1) \tag{22}$$

whose solution is given by

$$\widetilde{f}_h(x) := \begin{cases} \frac{1}{\sigma^2 \phi_\sigma(x)} \int_x^\infty h(t)\phi_\sigma(t)\, dt, & x \geq 1, \\ \frac{1}{\sigma^2 \phi_\sigma(x)} \int_{-\infty}^x h(t)\phi_\sigma(t)\, dt & x \leq -1. \end{cases} \tag{23}$$

Working with (23), it will be easier to treat the derivative functions. In fact, in (2), the integrand function $h - \Phi_\sigma(h)$ is centralized under the normal expectation. However, it is not the case when taking derivatives. This is one reason why we introduce (22). Note that in general, the right-hand side of (23) can not be extended as a continuous function on $\mathbb{R}$, except in the special case $\Phi_\sigma(h) = 0$ where we recover the solution of classical Stein's equation.

To study $\widetilde{f}_h$, we introduce the function space $\mathcal{E}_\sigma$: for any $\sigma > 0$, let $\mathcal{E}_\sigma$ be the space of all Borel functions h on $\mathbb{R} \setminus (-1, 1)$ such that $\int_{|x|\geq 1} |h(x)P(x)|\phi_\sigma(x)dx < \infty$ for any polynomial P. Note that $\mathcal{E}_\sigma$ is a vector space which contains all Laurent polynomials (that is, polynomials in x and x^{-1}) and is stable by multiplication by Laurent polynomials. Furthermore, as shown by the lemma below, it is invariant by the operator $h \to \widetilde{f}_h$.

Lemma B.5 *Let $h \in \mathcal{E}_\sigma$. Then the function $\widetilde{f}_h$ is well defined and $\widetilde{f}_h \in \mathcal{E}_\sigma$. Furthermore, if H is a primitive function of h, then $H \in \mathcal{E}_\sigma$.*

Proof. Let P be an arbitrary polynomial on $\mathbb{R}$. Then

$$\int_1^\infty |P(x)\widetilde{f}_h(x)|\phi_\sigma(x)\, dx \leq \frac{1}{\sigma^2} \int_1^\infty dx\, |P(x)| \int_x^\infty |h(t)|\phi_\sigma(t)\, dt = \frac{1}{\sigma^2} \int_1^\infty dt\, |h(t)|\phi_\sigma(t) \int_1^t |P(x)|\, dx.$$

There exists a polynomial Q such that $\int_1^t |P(x)|\, dx \leq Q(t)$ for any $t \geq 0$. Therefore, the fact that $h \in \mathcal{E}_\sigma$ implies that $\int_1^\infty |P(x)\widetilde{f}_h(x)|\phi_\sigma(x)\, dx < +\infty$. The finiteness of the integral on $(-\infty, 1]$ is similar. The second assertion can be proved by integration by part. □

Remark B.6 Note that $\widetilde{f}_h$ is the only solution of (22) in $\mathcal{E}_\sigma$, provided that $h \in \mathcal{E}_\sigma$.

More generally, for the derivatives of $\widetilde{f}_h$, we consider, for any integer $N \geq 1$, the set $\mathcal{E}_\sigma^N$ which contains all functions h such that h is N times differentiable on $\mathbb{R} \setminus (-1, 1)$ and that $h^{(N)} \in \mathcal{E}_\sigma$. It is not difficult to observe that $h \in \mathcal{E}_\sigma^N$ if and only if it is a primitive function of an element in $\mathcal{E}_\sigma^{N-1}$. The relationship between $\mathcal{E}_\sigma^N$ and $\mathcal{H}_{\alpha,p}^N$ is as follows.

Lemma B.7 *If $h \in \mathcal{H}^N_{\alpha,p}$, then the restriction of h on $\mathbb{R} \setminus (-1, 1)$ is in $\mathcal{E}^N_\sigma$.*

Proof. It suffices to show that the restriction of $h^{(N)}$ on $\mathbb{R} \setminus (-1, 1)$ is in $\mathcal{E}_\sigma$. This is obvious since $h_c^{(N)}$ has at most polynomial increasing speed at infinity. □

Definition B.8 For any derivable function h on $\mathbb{R} \setminus (-1, 1)$, define the operator

$$\Lambda(h)(x) := \Big(\frac{h(x)}{x}\Big)'. \tag{24}$$

Lemma B.9 *If $h \in \mathcal{E}^1_\sigma$, then $\Lambda(h) \in \mathcal{E}_\sigma$. Furthermore, we have the following equality:*

$$\widetilde{f}'_h(x) = x\widetilde{f}_{\Lambda(h)}(x). \tag{25}$$

Proof. By definition, $\Lambda(h)(x) = h'(x)/x - h(x)/x^2$, so $\Lambda(h) \in \mathcal{E}_\sigma$. To prove the equality, it suffices to verify that the function $u(x) := x^{-1}\widetilde{f}'_h(x)$ satisfies the equation (22) for $\Lambda(h)$ (see Remark above). In fact, if we divide the both side of the equation $x\widetilde{f}_h(x) - \sigma^2\widetilde{f}'_h(x) = h(x)$ by x and then take the derivative, we obtain $xu(x) - \sigma^2 u'(x) = \Lambda(h)(x)$. □

Lemma B.10 *If $h \in \mathcal{E}_\sigma$ and if l is a real number such that $h(x) = O(|x|^l)$, then $\widetilde{f}_h(x) = O(|x|^{l-1})$.*

Proof. Recall that ([5, LemA.1]) if $|h(x)| \le g(x)$ and if $g(x)/|x|$ is decreasing when $x > 0$ and is increasing when $x < 0$, then $|\widetilde{f}_h(x)| \le g(x)/|x|$. Hence, we prove the lemma for the cases where $l < 1$. By Lemma B.9, one has

$$\begin{aligned}\widetilde{f}_{|x|^l}(x) &= x^{-1}(|x|^l + \sigma^2\widetilde{f}'_{|x|^l}(x)) = \mathrm{sgn}(x)|x|^{l-1} + \sigma^2\widetilde{f}_{\Lambda(|x|^l)}(x)\\ &= \mathrm{sgn}(x)|x|^{l-1} + \sigma^2(l-1)\widetilde{f}_{|x|^{l-2}}(x).\end{aligned}$$

Thus, $\widetilde{f}_{|x|^{l-2}} = O(|x|^{l-3})$ implies $\widetilde{f}_{|x|^l} = O(|x|^{l-1})$. Hence by induction on l, we obtain the result. □

Remark B.11 With the notation of Barbour [1], the equivalent expectation form of $\widetilde{f}_h$ is given by

$$\widetilde{f}_h(x) = \begin{cases} \frac{\sqrt{2\pi}}{\sigma}\mathbb{E}\big[h(Z+x)e^{-\frac{Zx}{\sigma^2}}1\!\!1_{\{Z>0\}}\big], & x > 0\\ -\frac{\sqrt{2\pi}}{\sigma}\mathbb{E}\big[h(Z+x)e^{-\frac{Zx}{\sigma^2}}1\!\!1_{\{Z<0\}}\big], & x < 0\end{cases} \tag{26}$$

where $Z \sim N(0, \sigma^2)$. So the above lemma can be interpreted as : the function

$$\frac{1}{x^l}\mathbb{E}\big[1\!\!1_{\{Z>0\}}(Z+x)^{l+1}e^{-\frac{Zx}{\sigma^2}}\big]$$

is bounded on $[1,+\infty)$. We can then deduce easily the following assertion : for all $l \in \mathbb{R}$ and $m \in \mathbb{R}_+$, the function

$$\frac{1}{x^l}\mathbb{E}\big[\mathbb{1}_{\{Z>0\}}(Zx)^m(Z+x)^{l+1}e^{-\frac{Zx}{\sigma^2}}\big]$$

is bounded on $[1,+\infty)$ by using the fact that the function $u^m e^{-\frac{u}{2\sigma^2}}$ is bounded on $[0,\infty)$.

We give below the relationship between the derivatives of $\widetilde{f}_h$ and of h. In the following two formulas, the first one computes $\widetilde{f}_h^{(N)}$ using the operator (24) and the second one expresses $\Lambda^N(h)$ using derivatives of h. Their proofs are by induction, which we omit in this article (interest readers may refer to [10, p.144-145]). We only remind that the first formula is a generalization of (25).

Lemma B.12 *If $h \in \mathcal{E}_\sigma^N$ with N being a strictly positive integer, then*

$$\widetilde{f}_h^{(N)}(x) = \sum_{k=0}^{\lfloor N/2\rfloor}\binom{N}{2k}(2k-1)!!x^{N-2k}\widetilde{f}_{\Lambda^{N-k}(h)}(x); \tag{27}$$

$$\Lambda^N(h)(x) = \sum_{k=0}^{N}(-1)^k(2k-1)!!\binom{N+k}{2k}\frac{h^{(N-k)}(x)}{x^{N+k}}. \tag{28}$$

where we have used the convention $(-1)!! = 1$ and $\lfloor N/2\rfloor$ denotes the largest integer not exceeding $N/2$.

Remark B.13 1) For any function $h \in \mathcal{E}_\sigma^N$, the above results also hold for $\widetilde{f}_h^{(m)}(x)$ and $\Lambda^m(h)$ where $1 \leq m \leq N$. As the operator $h \to \widetilde{f}_h$ is linear on h, the above lemma enables us to write the derivatives of $\widetilde{f}_h$ as a linear combination of derivatives of h with Laurent polynomial coefficients and then to deduce their increasing speed at infinity.

2) The derivative function $\widetilde{f}_h^{(N+1)}$ has to be treated differently. In fact, we can no longer apply (28) to $\Lambda^{N+1}(h)$ since $h^{(N+1)}$ does not necessarily exist. We separate the first term where $k = 0$ in (27) from the others and then take the derivative to obtain

$$\begin{aligned}\widetilde{f}_h^{(N+1)} &= x^N\widetilde{f}'_{\Lambda^N(h)}(x) + \sum_{k=0}^{\lfloor N/2\rfloor}\binom{N}{2k}(2k-1)!!(N-2k)x^{N-2k-1}\widetilde{f}_{\Lambda^{N-k}(h)}(x)\\ &\quad + \sum_{k=1}^{\lfloor N/2\rfloor}\binom{N}{2k}(2k-1)!!x^{N-2k}\widetilde{f}'_{\Lambda^{N-k}(h)}(x)\\ &= x^N\widetilde{f}'_{\Lambda^N(h)}(x) + \sum_{k=1}^{\lfloor (N+1)/2\rfloor}\binom{N+1}{2k}(2k-1)!!x^{N+1-2k}\widetilde{f}_{\Lambda^{N+1-k}(h)}(x).\end{aligned}$$

This will be a crucial point in the proof of Proposition B.15.

Lemma B.14 *Let h be a Borel function defined on $A_2 \cup A_3 = \mathbb{R}\setminus(-1,1)$. If $\|h\|_{\alpha,p}^{A} < +\infty$ where $A = A_2$ or A_3, then, for any integer $n \geqslant 0$, one has*

$$\|x^{n+1}\widetilde{f}_{h/x^n}\|_{\alpha,p}^{A} < +\infty, \qquad \|x^n\widetilde{f}'_{h/x^n}\|_{\alpha,p}^{A} < +\infty.$$

Proof. We only prove for A_2 and the case for A_3 is by symmetry. Let $g(x) = h(x)/x^n$. Assume that x and y are two real numbers such that $1 \leqslant x < y$. Then one has

$$\frac{|x^{n+1}\widetilde{f}_g(x) - y^{n+1}\widetilde{f}_g(y)|}{|x-y|^{\alpha}(1+|x|^p+|y|^p)} = \frac{\sqrt{2\pi}}{\sigma}\mathbb{E}\Big[I_{\{Z>0\}}\frac{\Big|\frac{h(Z+x)x^{n+1}}{(Z+x)^n}e^{-Zx/\sigma^2} - \frac{h(Z+y)y^{n+1}}{(Z+y)^n}e^{-Zy/\sigma^2}\Big|}{|x-y|^{\alpha}(1+|x|^p+|y|^p)}\Big],$$

which can be bounded from above by the sum of the following two terms

$$\frac{\sqrt{2\pi}}{\sigma}\mathbb{E}\Big[I_{\{Z>0\}}\frac{|h(Z+x)-h(Z+y)|}{|x-y|^{\alpha}(1+|x|^p+|y|^p)}\cdot\frac{y^{n+1}}{(Z+y)^n}e^{-\frac{Zy}{\sigma^2}}\Big] \tag{29}$$

$$\frac{\sqrt{2\pi}}{\sigma}\mathbb{E}\Big[I_{\{Z>0\}}|h(Z+x)|\frac{\Big|\frac{x^{n+1}}{(Z+x)^n}e^{-\frac{Zx}{\sigma^2}} - \frac{y^{n+1}}{(Z+y)^n}e^{-\frac{Zy}{\sigma^2}}\Big|}{|x-y|^{\alpha}(1+|x|^p+|y|^p)}\Big] \tag{30}$$

Note that (29) is bounded from above by

$$\|h\|_{\alpha,p}^{A_2}\frac{\sqrt{2\pi}}{\sigma}y^{n+1}\mathbb{E}\Big[I_{\{Z>0\}}\frac{1}{(Z+y)^n}e^{-\frac{Zy}{\sigma^2}}\Big] = \|h\|_{\alpha,p}^{A_2}\,y^{n+1}\widetilde{f}_{\frac{1}{|x|^n}}(y).$$

By Lemma B.10, this quantity is bounded. We then consider the upper bound of (30) under the supplementary condition that $y \leqslant 2x$. As $\|h\|_{\alpha,p}^{A_2} < +\infty$, there exists a constant $C > 0$ such that $h(x) \leqslant C|x|^{\alpha+p}$. By applying the mean value theorem on the function $\frac{x^{n+1}}{(Z+x)^n}e^{-\frac{Zx}{\sigma^2}}$ and the fact that $|x-y|^{\alpha-1}(1+|x|^p+|y|^p) \geq |x|^{\alpha-1+p}$ where $\alpha < 1$,

$$\begin{aligned}
(30) &\leqslant \frac{\sqrt{2\pi}}{\sigma}|x|^{1-\alpha-p}\mathbb{E}\Big[I_{\{Z>0\}}C(Z+x)^{\alpha+p}e^{-\frac{Zx}{\sigma^2}}\Big(\frac{(n+1)(2x)^n}{(Z+x)^n} + \frac{n(2x)^{n+1}}{(Z+x)^{n+1}} + \frac{Z(2x)^{n+1}}{\sigma^2(Z+x)^n}\Big)\Big] \\
&\leqslant C\frac{\sqrt{2\pi}}{\sigma}|x|^{1-\alpha-p}\Big\{(2^n\cdot(n+1) + 2^{n+1}\cdot n)\mathbb{E}\big[I_{\{Z>0\}}(Z+x)^{\alpha+p}e^{-\frac{Zx}{\sigma^2}}\big] \\
&\qquad + 2^{n+1}\mathbb{E}\Big[I_{\{Z>0\}}\frac{Zx}{\sigma^2}(Z+x)^{\alpha+p}e^{-\frac{Zx}{\sigma^2}}\Big]\Big\} \\
&\ll |x|^{1-\alpha-p}\Big\{E\big[I_{\{Z>0\}}(Z+x)^{\alpha+p}e^{-\frac{Zx}{\sigma^2}}\big] + \mathbb{E}\Big[I_{\{Z>0\}}\frac{Zx}{\sigma^2}(Z+x)^{\alpha+p}e^{-\frac{Zx}{\sigma^2}}\Big]\Big\},
\end{aligned}$$

which is bounded (see Remark B.11). In the case where $y > 2x$, one has $x^{n+1}/(Z+x)^n \leq x$ when $Z \geq 0$, and $|x-y|^{\alpha}(1+|x|^p+|y|^p) \geq \max(|x|^{\alpha+p}, \left(\frac{y}{2}\right)^{\alpha}\cdot y^p)$, so

$$(30) \leqslant C\frac{\sqrt{2\pi}}{\sigma}\Big\{\frac{1}{2|x|^{\alpha+p-1}}\mathbb{E}\Big[I_{\{Z>0\}}(Z+x)^{\alpha+p}e^{-\frac{Zx}{\sigma^2}}\Big] + \frac{2^{\alpha}}{|y|^{\alpha+p-1}}\mathbb{E}\Big[I_{\{Z>0\}}(Z+y)^{\alpha+p}e^{-\frac{Zy}{\sigma^2}}\Big]\Big\},$$

which is bounded. □

We now give the final part of the proof.

Proposition B.15 *Let $N \geqslant 0$ be an integer, $\alpha \in (0,1]$ and $p \geqslant 0$. Let h be a function defined on $A_2 \cup A_3 = \mathbb{R} \setminus (-1,1)$ which is N times differentiable and such that $h^{(N)}$ is locally of finite variation, having finitely many jumps and verifying $\|h_c^{(N)}\|_{\alpha,p}^{A} < +\infty$ where $A = A_2$ or A_3. Then the function $\widetilde{f}_h$ is $N+1$ times differentiable, $\widetilde{f}_h^{(N+1)}$ is locally of finite variation, having finitely many jumps and verifying $\|\widetilde{f}_{h,c}^{(N+1)}\|_{\alpha,p}^{A} < +\infty$.*

Proof. The function $\widetilde{f}_h$ is $N+1$ times differentiable by (20), $\widetilde{f}_h^{(N+1)}$ is locally of finite variation, having only finitely many jumps. By virtue of Lemmas B.3 and B.4, it suffices to prove

$$\max\{\|\widetilde{f}_{h,c}^{(N+1)}\|_{\alpha,p}^{(-\infty,-b]}, \|\widetilde{f}_{h,c}^{(N+1)}\|_{\alpha,p}^{[b,+\infty)}\} < +\infty$$

for sufficiently positive number b. Therefore, without loss of generality, we may assume that $h^{(N)}$ is continuous and hence $\widetilde{f}_h^{(N+1)}$ is also continuous.
By Remark B.13 2), the function $\widetilde{f}_h^{(N+1)}$ can be written as a linear combination of $x^N \widetilde{f}'_{\Lambda^N(h)}$ and terms of the form $x^{N+1-2k}\widetilde{f}_{\Lambda^{N+1-k}(h)}(x)$ where $k = 1, \cdots, \lfloor \frac{N+1}{2} \rfloor$. By (28), $x^N \widetilde{f}'_{\Lambda^N(h)}$ itself is also a linear combination of $x^N \widetilde{f}'_{h^{(N-i)}/x^{N+i}}$ where $i = 0, \cdots, N$. As $\|h^{(N)}\|_{\alpha,p}^{A} < \infty$, we have, similar as in in Lemma 3.1 4), that $\|h^{(N-i)}\|_{\alpha,p+i}^{A} < +\infty$. Hence $\|h^{(N-i)}/x^i\|_{\alpha,p}^{A} < \infty$ by Lemma B.2 and Lemma B.14 then implies that $\|x^N \widetilde{f}'_{h^{(N-i)}/x^{N+i}}\|_{\alpha,p}^{A} < \infty$.
The terms $x^{N+1-2k}\widetilde{f}_{\Lambda^{N+1-k}(h)}(x)$ are also, by (28) again, linear combinations of the functions of the form $x^{N+1-2k}\widetilde{f}_{h^{(N+1-k-i)}/x^{N+1-k+i}}$. By a similar argument as above using Lemma B.2, $\|h^{(N+1-k-i)}/x^{1+k+i}\|_{\alpha,p}^{A} < \infty$. Finally, we apply Lemma B.14 to obtain $\|x^{N+1-2k}\widetilde{f}_{h^{(N+1-k-i)}/x^{N+1-k+i}}\|_{\alpha,p}^{A} < \infty$, which completes the proof.

□

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